Setting $n = 2$, we obtain $c_{31^2} = 1$, since $(6) = \Gamma(-4, 6)$ vanishes, as do all the terms on the right save (31^2) and (31^3), which are $-\Gamma(2)$ and $\Gamma(2)$, respectively. Setting $n = 3$, we obtain, similarly, $c_{41} = 1$.

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*AN APPLICATION OF ALGEBRAIC TOPOLOGY TO NUMERICAL ANALYSIS: ON THE EXISTENCE OF A SOLUTION TO THE NETWORK PROBLEM**

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The electrical network problem was first treated comprehensively by Kirchhoff[1] in 1847. A complete proof for the existence of a solution was given by Hermann Weyl[2] in 1923 for the case of a purely resistive network when sources of electromotive force are placed in series with the branches. This proof was elaborated by Eckmann[3] in 1945. A condition for the existence of a solution given by Synge,[4] attempting to cover a more general case, was unfortunately incorrect, as simple counterexamples show. In this paper we show the existence and uniqueness of a solution to the network problem under a condition which amounts to assuming that the dissipative power is positive definite. Since this condition is satisfied by any physically realizable network, it may be said that this result covers the general physical case (steady state). Actually, we state the electrical network problem in a purely algebraic-topological way. This is of interest since "electrical" networks are used to solve certain ordinary and partial differential equations. The results of this paper are essential to the proof of the validity of Kron's method of tearing,[5] established in a second paper.[6] Our first task is to describe a mathematical model for the quantities and relations which exist in an electrical network.

Let K be an electrical network: We shall consider K as an oriented one-dimensional complex. A set of currents flowing through the branches of K may be considered as the assigning of a complex number to each branch ("coil," in Kron's terminology). Hence such a set of currents will be treated as a vector or oriented 1-chain. The space of such sets of branch currents thus coincides with the group $C^1(K)$ of oriented 1-chains over the coefficient field of complex numbers. A mesh current, the current flowing around an oriented closed loop, corresponds to a 1-cycle, but, as we shall see, it is more appropriate to identify it with an element of the first homology group $H^1(K)$ of K. In fact, the space of such loop currents (they are also called "mesh currents") is isomorphic with $H^1(K)$. With these identifications, the transformation $i = Ci'$ used by Kron[7] is the natural homomorphism[8] of $H^1(K)$ into $C^1(K)$. Here i', i are column vectors representing elements of $H^1(K)$, $C^1(K)$ with respect to bases chosen for these spaces. We recall, also, that $C^1(K)$ may be regarded as the relative homology group of K modulo its zero skeleton. Physically this transformation may be considered as an expression of Kirchhoff's current law. It can also be seen that the space of node currents (node = vertex) is isomorphic to the group $P^0(K)$ of bounding zero chains. In Kron's notation the boundary operator is expressed by the equation $I' = A_tJ$, where J and I' are column vectors representing vectors in $C^1(K)$ and $P^0(K)$ with respect to bases chosen for these spaces. We shall use the notation more customary in topology, $I' = \partial J$, ∂ being the boundary operator. In this context the boundary operator may be interpreted as an alternative expression of Kirchhoff's current law, equivalent to the above statement of the same law. We may combine the mappings C and ∂ to form the following "reduced" homology sequence of the pair (K, K^0), K^0 being the zero skeleton of K:

$$0 \rightarrow H^1(K) \xrightarrow{C} C^1(K) \xrightarrow{\partial} P^0(K) \rightarrow 0.$$

The result $\partial C = 0$ (or $A_tC = 0$) is a consequence of the exactness of this sequence and is referred to by Kron as "the orthogonality condition." Of course this result can also be proved directly very easily.

The dual set of relations involved in considering the spaces of voltages and potentials will next be identified with the corresponding "reduced" cohomology sequence. We start at the opposite end of the sequence. First we identify the space of node potentials with the group $C_0(K)$ of zero cochains of K. The space of potential differences (node-pair potentials) coincides with the subgroup $P_0(K)$ of $C_0(K)$, dual to $P^0(K)$, isomorphic to $C_0(K)$ modulo the subgroup of 0-cocycles. $P_0(K)$ is selected as the image of $P^0(K)$ under the following isomorphism: Let $C^0 = \sum a_i\sigma_i^0$ be an oriented 0-chain of K, the σ_i^0 being oriented 0-cells of K, the summation running over the set of all 0-cells of K, and the a_i being complex numbers. Let $\varphi(C^0)$ be the 0-cochain f such that $f(\sigma_i) = a_i$. The transformation φ is an isomorphism of $C^\circ(K)$ on $C_0(K)$. Let $P_0(K)$ be the image of $P^0(K)$ under φ. Also, if $\{b_k\}$ is a basis for $P^0(K)$, we shall call the set $\{\varphi(b_k)\}$ of images the "same" basis for $P_0(K)$. Now the space of branch voltages ("coil voltages" in Kron's terminology) may be identified with the group of 1-cochains of K, $C_1(K)$. The coboundary operator $E = \delta E'$ (or $E = AE'$, in Kron's notation), where E, E' represent elements of $C_1(K)$, $P_0(K)$, can be considered as a statement of Kirchhoff's voltage law. Finally, the space of mesh emfs coincides with the first cohomology group $H_1(K)$. Again, if we pick fixed bases for $H^1(K)$ and $C^1(K)$ and write a matrix equation for the natural homomorphism C and then use the "same" basis to express the dual homomorphism of $C_1(K)$ onto $H_1(K)$, then the mapping takes the form used by Kron, $C_tV = \acute{e}$. This mapping is another equivalent statement of Kirchhoff's voltage law. These two mappings yield for us the following reduced cohomology sequence:

$$0 \leftarrow H_1(K) \xleftarrow{C_t} C_1(K) \xleftarrow{\delta} P_0(K) \leftarrow 0.$$

By exactness, $C_t\delta = 0$ (or $C_tA = 0$); this condition is termed by Kron "the orthogonality of mesh emfs and of node-pair potentials."

The Twisted Isomorphism.—When steady-state conditions prevail, the differential equations describing the various branch currents and branch voltages in the network may be reduced to a matrix equation $V = LJ$, a single equation for the entire network, where J is a column vector representing a 1-chain with respect to a basis chosen for $C^1(K)$ and V is a column vector representing a 1-cochain with respect to the "same" basis for $C_1(K)$. Finally, L, called the "impedance matrix," is determined solely by the electromagnetic properties of the various coils (their resistances, mutual inductances, etc.); it is essential to realize that the transformation L is independent of the manner in which the branches are hooked together, i.e., independent of the topological structure of K. In Kron's notation L is denoted by Z.

The electrical network problem may be stated in physical terms as follows: Given a set of coils with given electromagnetic properties determining a transformation L, hooked together in a prescribed way to determine a network K and, given a set e' of sources of emf impressed in series with the meshes (e' is an element of $H_1(K)$) and a set I' of currents impressed from outside the system on the nodes of K (I' is an element of $P^0(K)$), the problem is to find the current flowing through and the potential drop across each coil—these subject to Kirchhoff's laws.

The "electrical" network problem may be stated in purely mathematical terms as follows: Given a complex K (determining the transformations C and ∂) and the matrix L, given $e' \in H_1(K)$, $I' \in P^0(K)$, the problem is to find $V \in C_1(K)$ and $J \in C^1(K)$ such that $1^1\ V = LJ$, $2^0\ C_tV = e'$ and $3^\circ\ \partial J = I'$.

Whether or not a solution to the network problem exists depends, of course, upon the nature of L. We shall say that a matrix L and the transformation represented by L is *power definite* if $L + \bar{L}_t$ is positive definite. The dissipative power $1/2(V_t\bar{J} + \bar{V}_tJ)$ is positive definite if and only if L is power definite. Hence, for any physically realizable network, this condition is satisfied.

THEOREM. *If L is power definite, then the network problem has one and only one solution.*

The problem may be readily visualized by means of the following "algebraic network diagram":

$$
\begin{tikzcd}
0 \arrow[r] & H^1(K) \arrow[r, "C"] \arrow[d, "L'"'] & C^1(K) \arrow[r, "\partial"] \arrow[d, shift right, "L"'] & P^0(K) \arrow[r] & 0 \\
0 & H_1(K) \arrow[l] & C_1(K) \arrow[l, "C_t"'] \arrow[u, shift right, "Y"'] & P_0(K) \arrow[l, "\delta"'] \arrow[u, "Y'"'] & 0 \arrow[l]
\end{tikzcd}
$$

Here Y is the inverse of L, $L' = C_tLC$, $Y' = \partial Y\delta$. We will show that under the conditions on L the mappings L' and Y' are one to one, that is, L' and Y' are nonsingular. We will then exhibit the solution. For suppose that $L'i' = 0$, and let $i = Ci'$. Then $\bar{i}_t'L'i'$, which equals $\bar{i}_tLi$, is zero, and so, too, is $\bar{i}_t\bar{L}_ti$ zero. Thus $\bar{i}_t(L + \bar{L}_t)i = 0$, and so, by the power-definiteness assumption on L, i must be zero. But since, by exactness, C is an isomorphism into, i' must also be zero. Thus the kernel of L' is zero, and, since the spaces have the same dimension, L' must be one to one. By a similar argument, Y' is also shown to have an inverse. (By exactly the same argument one may show that L' and Y' have inverses if $L - \bar{L}_t$ is assumed to be definite.)

One may easily verify that $J = C(L')^{-1}e' + Y\delta(Y')^{-1}I'$ and $V = LJ$ will constitute a solution. Next we consider the question of uniqueness.

Suppose, then, that V, J and V^*, J^* are solutions to the given problem. Let $V\# = V - V^*$ and $J\# = J - J^*$. Then $\partial J\# = 0$, and so, by the exactness of the homology sequence, there is an $i\#$ in $H^1(K)$ such that $J\# = Ci\#$. But $V\# = LJ\#$, and, since $C_tV\# = 0$, we have $C_tLCi\# = 0$. By the above proof, $i\#$ and hence $J\#$ must be zero, so that $J = J^*$ and $V = V^*$.

Of course, to solve the network problem, to obtain the general solution, it is not necessary to invert both L' and Y'. To solve, for example, by means of inverting L' only, we proceed as follows: Let $C^1(K)$ be written as the direct sum of the image $M^1(K)$ of $H^1(K)$ under C ($M^1(K)$ is the group of cycles) plus another group $P^1(K)$ (cf. companion paper,[6] Sec. II). Then J, an element of $C^1(K)$, can be written as $J = i + I$, $i \in M^1(K)$, $I \in P^1(K)$. Thus I is chosen such that $\partial I = I'$ and may be considered as known. Then $V = L(i + I)$. Now $e' = C_tV = C_tL(i + I)$. But i may be written as the image of an element of $H^1(K)$, $i = Ci'$, $i' \in H^1(K)$. Thus $C_tLCi' = e' - C_tLI$, so that $i' = (L')^{-1}(e' - C_tLI)$. Thus J is determined, $J = Ci' + I$, and $V = LJ$, so that the solution is obtained by inverting L' only. Similarly, one may obtain the solution by inverting Y'. We also have the following consequence of the above theorem.

COROLLARY. *If a network problem is physically realizable, then there exists one and only one solution.*

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